Solutions to HW#13

1. $f(x,y) = 2x^2 + xy - 4y^2$, $\quad \nabla f(x,y) = (4x+y,\ x-8y) = (0,0)$

$$4x+y=0 \qquad \begin{pmatrix} 4 & 1 \\ 1 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$x-8y=0$$

Because $\begin{pmatrix} 4 & 1 \\ 1 & -8 \end{pmatrix}$ is invertible, the only solution is $(0,0)$.

2. $f(x,y) = xy(x+y-1)$

$$\nabla f(x,y) = \big(y(x+y-1)+xy,\ x(x+y-1)+xy\big) = (0,0)$$
$$y(2x+y-1) = 0$$
$$x(x+2y-1) = 0$$

$x=0$

- $y=0$: $\boxed{(0,0)}$
- $y\neq 0$: $y(y-1)=0$, $y=1$, $\boxed{(0,1)}$

$x \neq 0$

- $y=0$: $x(x-1)$, $x=1$, $\boxed{(1,0)}$
- $y \neq 0$:

$$2x+y-1=0$$
$$x+2y-1=0$$
$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} =$$
$$= \frac{1}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\boxed{\left(\tfrac{1}{3},\ \tfrac{1}{3}\right)}$$

3. $f(x,y) = (5x-2y)^2$

Observe that $f(x,y) \geq 0$ with equality whenever $5x-2y=0$.
Thus, the set of critical points is $C = \{t(2,5) : t \in \mathbb{R}\}$

4. $f(x,y) = (x^2-1)(y^2-1)$

$\nabla f(x,y) = \left(2x(y^2-1),\ 2y(x^2-1)\right) = (0,0)$

$$2x(y^2-1) = 0 \qquad x=0 \quad y=\pm 1$$
$$2y(x^2-1) = 0 \qquad y=0 \quad x=\pm 1$$

Thus, the critical points are $(0,0), (1,1), (-1,1), \& (1,-1)$.

5. $f(x,y) = xy^{2/3}$. $\nabla f(x,y) = \left(y^{2/3}, \frac{2}{3}xy^{-1/3}\right)$.

Notice that the gradient is undefined whenever $y=0$.
Thus any point $(x,0)$ is a critical point. (Can you classify these critical points as maxima, minima, or saddle?).

6. $f(x,y) = \cos x \sin y$. Observe that $-1 \leq f(x,y) \leq 1$.

Thus f attains its maximum whenever $\cos x = 1$ and $\sin y = 1$.
Or, whenever $(x,y) = (m \cdot 2\pi, n2\pi + \frac{\pi}{2})$ for $m,n \in \mathbb{Z}$.
Similarly, f attains its minimum whenever $\cos x = -1$ and $\sin y = 1$ or $\cos x = 1$ and $\sin y = -1$.

7. $f(x,y) = e^{-x^2-y^2}$. Observe that f is a surface of revolution.

The critical points of f may thus be obtained from single-variable function $g(x) = e^{-x^2}$. Since $g'(x) = 0$ iff $x=0$, the critical point of f is $(0,0)$. At this point, f attains a maximum.

8. $f(x,y) = x^2+y^2+2x-y+3$.

$\nabla f(x,y) = (2x+2,\ 2y-1) = (0,0)$

Hence the only critical point is $(-1, \frac{1}{2})$.

$H_{f(-1,\frac{1}{2})} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ which is positive-definite.

It follows that f has a local min at $(-1, \frac{1}{2})$. This is also a global min of f.

Remark: You can also find and classify the critical points of f by completing the square and noting that f is a paraboloid.

9. $f(x,y) = -3x^2+6xy+2y^2+12x-12y$

$\nabla f(x,y) = (-6x+6y+12,\ 6x+4y-12) = (0,0)$

$$+\begin{cases} -6x+6y+12=0 \\ 6x+4y-12=0 \end{cases}$$

$$10y = 0 \Rightarrow y = 0.$$

Plugging into the first equation, we obtain $-6x+12=0$ or $x=2$.

Thus $(2,0)$ is the only critical point.

$H_{f(2,0)} = \begin{pmatrix} -6 & 6 \\ 6 & 4 \end{pmatrix}$ $\det(-6) < 0$, $\det(H_{f(2,0)}) < 0$.

Thus $(2,0)$ is a saddle point.

10. $f(x,y) = x^3+xy^2-y^2-x$

$\nabla f(x,y) = (3x^2+y^2-1,\ 2xy-2y) = (0,0)$

$$(1)\ 3x^2+y^2-1=0$$

$$(2)\ 2xy-2y=0$$

Equation (2) implies that $(x-1)y=0$, which is true iff $x=1$ or $y=0$.

If $y=0$, equation (1) becomes $3x^2-1=0$ or $x=\pm\frac{1}{\sqrt{3}}$.

If $x=1$, equation (1) becomes $3+y^2-1=0$, which has no real solutions.

Thus, the only critical points are $(\frac{-1}{\sqrt{3}},0)$ and $(\frac{1}{\sqrt{3}},0)$.

$$H_{g(x,y)} = \begin{pmatrix} 6x & 2y \\ 2y & 2x-2 \end{pmatrix}$$

$$H_{g(\frac{-1}{\sqrt{3}},0)} = \begin{pmatrix} \frac{-6}{\sqrt{3}} & 0 \\ 0 & \frac{-2}{\sqrt{3}}-2 \end{pmatrix};\ \det\left(\frac{-6}{\sqrt{3}}\right)<0,\ \det\left(H_{g(\frac{-1}{\sqrt{3}},0)}\right)>0$$

so $(\frac{-1}{\sqrt{3}},0)$ is a local max.

$$H_{g(\frac{1}{\sqrt{3}},0)} = \begin{pmatrix} \frac{6}{\sqrt{3}} & 0 \\ 0 & \frac{2}{\sqrt{3}}-2 \end{pmatrix};\ \det\left(\frac{6}{\sqrt{3}}\right)>0,\ \det\left(H_{g(\frac{1}{\sqrt{3}},0)}\right)<0.$$

so $(\frac{1}{\sqrt{3}},0)$ is a saddle point.

11. $f(x,y) = (x^2-x)(y^2-y)$

$$\nabla f(x,y) = \big((2x-1)(y^2-y),\ (x^2-x)(2y-1)\big) = (0,0)$$

$$\begin{cases} (2x-1)\,y(y-1) = 0 \\ x(x-1)(2y-1) = 0 \end{cases}$$

- $\underline{x=0}$: $-(y-1)y=0$
  - $\underline{y=0}$: $\boxed{(0,0)}$
  - $\underline{y=1}$: $\boxed{(0,1)}$
- $\underline{x\neq 0}$: $(x-1)(2y-1)=0$
  - $\underline{x=1}$: $y^2-y=0$
    - $\underline{y=0}$: $\boxed{(1,0)}$
    - $\underline{y=1}$: $\boxed{(1,1)}$
  - $\underline{y=\frac{1}{2}}$: $-\frac{1}{4}(2x-1)=0$
    - $\underline{x=\frac{1}{2}}$: $\boxed{(\frac{1}{2},\frac{1}{2})}$

$$H_{f(x,y)} = \begin{pmatrix} 2(y^2-y) & (2x-1)(2y-1) \\ (2x-1)(2y-1) & 2(x^2-x) \end{pmatrix}$$

$$H_{f(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ Since $\det(0)=0$, the second derivative gives no information. However, if $\delta>0$ but $\delta<1$, $(x,y)\in B_\delta(0,0)$ $\Rightarrow |x|>x^2$ and $|y|>y^2$. So $(x^2-x)(y^2-y)>0$ whenever $x>0, y>0$ & $(x,y)\in B_\delta(0,0)$. On the other hand, if $x<0, y>0$ & $(x,y)\in B_\delta(0,0)$, $(x^2-x)(y^2-y)<0$. Thus $(0,0)$ is a saddle point.

$H_{f(0,1)} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Since $\det(\cdot)=0$, again there is no information. Since, for any $\delta>0$, it is possible to find points $(x,y) \in B_\delta(0,1)$ s.t. $x^2-x>0$ and $y^2-y>0$ and points of the form $x^2-x<0$ and $y^2-y>0$, $(0,1)$ is a saddle point. $(1,0)$ is also a saddle point by symmetry.

$H_{f(1,1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Again, second derivative gives no information. You should verify that $(1,1)$ is a saddle point.

$H_{f(1/2,1/2)} = \begin{pmatrix} -1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ which is positive-definite. It follows that $(1/2, 1/2)$ is a local max.

12. $f(x,y) = x^4 - 3x^2y^2 + y^4$

$\nabla f(x,y) = (4x^3 - 6xy^2, -6x^2y + 4y^3) = (0,0)$

$$\begin{cases} 2x^3 - 3xy^2 = 0 \\ -3x^2y + 2y^3 = 0 \end{cases}$$

$\underline{x=0}$

$\Downarrow$

$\underline{y=0}$

$\boxed{(0,0)}$

$\underline{x \neq 0}$

$$\begin{cases} 2x^2 - 3y^2 = 0 \\ -3x^2y + 2y^3 = 0 \end{cases}$$

$\underline{y=0}$

$\Downarrow$

$\underline{x=0}$

$\boxed{(0,0)}$

$\underline{y \neq 0}$

$$\begin{cases} 2x^2 - 3y^2 = 0 \\ -3x^2 + 2y = 0 \end{cases}$$

$$\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} x^2 \\ y^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\boxed{(0,0)}$ $\times$

Thus the only critical point is $(0,0)$.

$$H_{f(x,y)} = \begin{pmatrix} 12x^2 & -12xy \\ -12xy & 12y^2 \end{pmatrix} \text{ so } H_{f(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In other words, the second derivative tells us nothing.

Observe that $f(x,y) = x^4 - 3x^2y^2 + y^4 = x^4 - 2(xy)^2 + y^4 - (xy)^2 =$
$= (x^2-y^2) - (xy)^2 \equiv r^4\cos^2 2\theta - \frac{r^4}{4}\sin^2 2\theta.$
when $\theta = \frac{\pi}{4}$, we have $\frac{-r^4}{4}\sin^2 2\theta < 0$. On the other hand, when $\theta = 0$, we have $r^4\cos^2 2\theta > 0$ Thus f has a saddle at $(0,0)$ (why?).

13. $f(x,y) = (x^2+y^2)e^{-y}$

$$\nabla f(x,y) = \left(2xe^{-y},\ 2ye^{-y} - (x^2+y^2)e^{-y}\right) = (0,0)$$

$$\begin{cases} 2x = 0 \\ 2y - (x^2+y^2) = 0 \end{cases}$$

Our critical points are therefore $(0,0)$ and $(0,2)$.

$$H_{f(x,y)} = \begin{pmatrix} 2e^{-y} & -2xe^{-y} \\ -2xe^{-y} & (2-2y)e^{-y} - (2y - x^2 - y^2)e^{-y} \end{pmatrix}$$

$$H_{f(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$ Thus $(0,0)$ is a local min.

$$H_{f(0,2)} = \begin{pmatrix} 2e^{-2} & 0 \\ 0 & -2e^{-2} \end{pmatrix},$$ Thus $(0,2)$ is a saddle point.

14. $f(x,y) = x + y - \ln(xy) = x + y - \ln x - \ln y$

$$\nabla f(x,y) = \left(1 - \frac{1}{x}, 1 - \frac{1}{y}\right) = \left(\frac{x-1}{x}, \frac{y-1}{y}\right).$$

The gradient is undefined when $(x,y) = (0,0)$, However, this is not a critical point, since f is itself undefined on $(0,0)$.

$\nabla f(x,y) = (0,0)$ when $(x,y) = (1,1)$

$$H_{f(x,y)} = \begin{pmatrix} \frac{1}{x^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix} ; \quad H_{f(1,1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ so $(1,1)$ is a local min.

15. $f(x,y,z) = (x^2 + y^2 + z^2 + yz)$

$$\nabla f(x,y,z) = (2x, 2y + z, 2z + y) = (0,0,0)$$

$$\begin{cases} x = 0 \\ 2y + z = 0 \\ y + 2z = 0 \end{cases}$$

Solving this equation indicates that $(0,0,0)$ is the only critical point.

$$H_{f(x,y,z)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$ Since $\det(2) > 0$; $\det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} > 0$; $\det\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} > 0$

$(0,0,0)$ is therefore a local min.

16. $f(x,y) = x^2+y^2+z^2$ where $z = 4x+3y+1$ is the function that we wish to minimize.

$$\nabla f(x,y) = (2x + 2(4x+3y+1)\cdot 4,\ 2y + 2\cdot 3(4x+3y+1)) = (0,0).$$

$$\begin{cases} 15x + 12y = -4 \\ 12x + 10y = -3 \end{cases}$$

$$\begin{pmatrix} 15 & 12 \\ 12 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$$

Because $A = \begin{pmatrix} 15 & 12 \\ 12 & 10 \end{pmatrix}$ is invertible with inverse $A^{-1} =$ $= \frac{1}{150-144}\begin{pmatrix} 10 & -12 \\ -12 & 15 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 10 & -12 \\ -12 & 15 \end{pmatrix}$, the desired answer can be obtained by multiplying A^{-1} by $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$ (why?).

17. Finding the points on the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ is equivalent to minimizing $x^2+y^2+z^2$ where $x^2 = 1 - \left(\frac{y^2}{4} + \frac{z^2}{9}\right)$.

Let $f(y,z) = y^2+z^2+1-\left(\frac{y^2}{4}+\frac{z^2}{9}\right)$. Then

$$\nabla f(z,y) = \left(2y - \frac{y}{2},\ 2z - \frac{2z}{9}\right) = (0,0).$$

The only critical point is $(0,0)$.

$H_{f(0,0)} = \begin{pmatrix} 2-\frac{1}{2} & 0 \\ 0 & 2-\frac{2}{9} \end{pmatrix}$ which is positive-definite. It follows that when $y=0$ and $z=0$, the minimum is attained. At these values, $x = \pm 1$. It follows that the points on the ellipsoid that are closest to the origin are $(-1,0,0)$ and $(1,0,0)$, which is in agreement with intuition

18. $f(s) = (2s, 3s-1, -s+2)$
$g(t) = (t+1, -t+3, 4t+5)$

Let $h(s,t) = \|f(s) - g(t)\|$. h is the function that we wish to minimize. Since $0 \le h(s,t)$, we might as well minimize H, where $H(s,t) = h^2(s,t) = \|f(s) - g(t)\|^2 = (2s-t-1)^2 + (3s+t-4)^2 + (-s-4t-3)^2$.

$\nabla H(s,t) = \big(2(2s-t-1)\cdot 2 + 2(3s+t-4)\cdot 3 - 2(-s-4t-3),\ -2(2s-t-1) + 2(3s+t-4) - 2(-s-4t-3)\cdot 4\big) = (0,0)$

$$\begin{cases} 2(2s-t-1) + 3(3s+t-4) - (-s-4t-3) = 0 \\ -(2s-t-1) + (3s+t-4) - 4(-s-4t-3) = 0 \end{cases}$$

This system of equations simplifies to

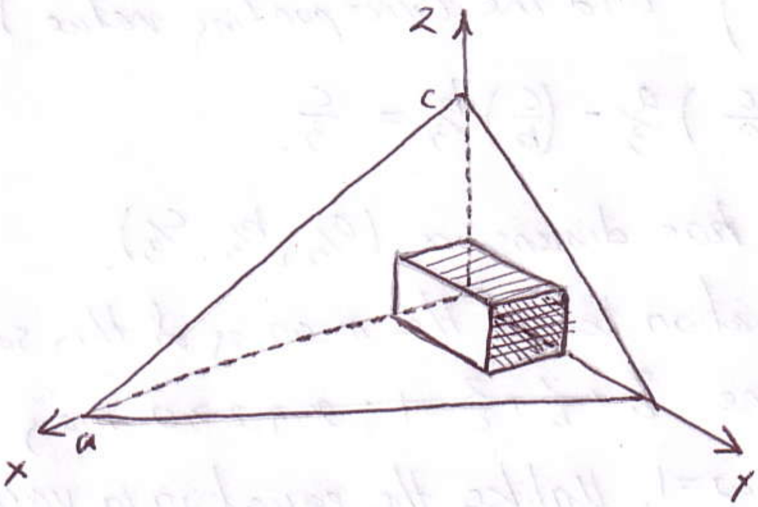
$$\begin{cases} 14s + 5t = 1 \\ 5s + 18t = -9 \end{cases} \quad \text{or} \quad \begin{pmatrix} 14 & 5 \\ 5 & 18 \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 11 \\ -9 \end{pmatrix}$$

Since the inverse matrix of the coefficient matrix of the equation is $\frac{1}{227}\begin{pmatrix} 18 & -5 \\ -5 & 14 \end{pmatrix}$, $\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 18 & -5 \\ -5 & 14 \end{pmatrix}\frac{1}{227}\begin{pmatrix} 11 \\ -9 \end{pmatrix}$ (why?)

Thus $s = \frac{243}{227}$ and $t = \frac{-181}{227}$. It follows that

$f\left(\frac{243}{227}\right)$ and $g\left(\frac{-181}{227}\right)$ are the desired pair.

19.



The volume of the box inscribed in the tetrahedron is

$V(x,y,z(x,y)) = xyz(x,y)$ where $z(x,y) = c - \left(\frac{c}{a}\right)x - \left(\frac{c}{b}\right)y$.

and $x>0, y>0$.

Thus, we would like to maximize $V(x,y) = cxy - \left(\frac{c}{a}\right)x^2y - \left(\frac{c}{b}\right)xy^2$.

Now $\nabla V(x,y) = \left(cy - \left(\frac{c}{a}\right)2xy - \left(\frac{c}{b}\right)y^2;\ cx - \left(\frac{c}{a}\right)x^2 - \left(\frac{c}{b}\right)2xy\right) =$

$(0,0)$ whenever

$$\begin{cases} aby - 2bxy - ay^2 = 0 \\ abx - 2axy - bx^2 = 0 \end{cases}$$

Since $x,y>0$, the above equations may further be reduced

to $\begin{cases} ab - 2bx - ay = 0 \\ ab - 2ay - bx = 0 \end{cases}$ or $\begin{cases} 2bx + ay = ab \\ bx + 2ay = ab \end{cases}$

or, in matrix form; $\begin{pmatrix} 2b & a \\ b & 2a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ab \\ ab \end{pmatrix}$

Thus $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3ab}\begin{pmatrix} 2a & -a \\ -b & 2b \end{pmatrix}\begin{pmatrix} ab \\ ab \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2a & -a \\ -b & 2b \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Notice that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{a}{3} \\ \frac{b}{3} \end{pmatrix}$ and the corresponding value for $z =$ $= z(a/3, b/3)$ is $c - \left(\frac{c}{a}\right)\frac{a}{3} - \left(\frac{c}{b}\right)\frac{b}{3} = \frac{c}{3}$.

Thus the biggest box has dimension $(a/3, b/3, c/3)$.

To get a deeper appreciation behind the meaning of this solution, observe that the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$; $x, y, z \geq 0$ may be written as $u + v + w = 1$. Unlike the equation in variables x, y, and z, the new equation is symmetric in the variables u, v, w.

The current problem, as it is originally stated, is to maximize $V(x,y,z) = xyz$ under the constraint $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

We can restate this problem in terms of new variables u, v, w:

$xyz = a\left(\frac{x}{a}\right) b\left(\frac{y}{b}\right) c\left(\frac{z}{c}\right) = abc\, uvw = V(u,v,w)$.

Notice that $V(u,v,w)$ is a symmetric function (i.e. $V(v,u,w) =$ $= V(u,w,v) = \ldots = V(u,v,w)$).

Because V is symmetric, any critical point (m,n,l) implies that (n,m,l), (l,m,n), (l,n,m), etc. are also critical points. Since $V(m,n,l) \leq V(M,M,M)$, where $M = \max\{m,n,l\}$, it is easy to see that maximum volume corresponds to a critical point (u,u,u) under the constraint $u+u+u=1$. (i.e. $u=v=w$).

Thus $3u = 1$ so $u = \frac{1}{3} \Rightarrow v = w = \frac{1}{3}$. $\Rightarrow \frac{x}{a} = \frac{1}{3}$, $\frac{y}{b} = \frac{1}{3}$, $\frac{z}{c} = \frac{1}{3}$.

so $(x,y,z) = (a/3, b/3, c/3)$.

What are the dimensions of a hyper-box (x,y,z,w) with volume $xyzw$ under the constraint $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} = 1$ if we wish to have the largest possible volume?

20. $xyz=1000$, $x,y,z>0$.

The cost of the box with dimensions (x,y,z) is therefore

$$C(x,y,z(x,y)) = 2\cdot 2xy + 2\cdot 4xz + 2\cdot 4yz =$$
$$= 2\left(2xy + \frac{4\cdot 1000}{y} + \frac{4\cdot 1000}{x}\right) = C(x,y).$$

Notice that $C(x,y)$ is symmetric so the minimum will occur when $x=y$ (if (a,b) is the x and y dimension of the minimum, then so is (b,a), but the cost is an increasing function of x, so $C(a,a) < C(b,a)$, which implies that (a,b) cannot be a critical point unless $a=b$).

Thus $C(x,y) \equiv C^*(x) = 2\left(2x^2 + \frac{8\cdot 1000}{x}\right)$.

$C^{*\prime}(x) = 0$ when $4x - \frac{8\cdot 1000}{x^2} = 0 \iff x^3 - 2000 = 0$

so $x = \sqrt[3]{2000} = 10\sqrt[3]{2} = y$. Therefore $z = \frac{1000}{xy} = \frac{1000}{100\sqrt[3]{4}} = \frac{10}{\sqrt[3]{4}}$

We can also solve this problem by appealing to the gradient.

$$\nabla C(x,y) = 2\left(2y - \frac{4\cdot 1000}{x^2},\ 2x - \frac{4\cdot 1000}{y^2}\right) = (0,0)$$

$$\begin{cases} y - \frac{2000}{x^2} = 0 \\ x - \frac{2000}{y^2} = 0 \end{cases} \text{ or } \begin{cases} x^2y - 2000 = 0 \\ xy^2 - 2000 = 0. \end{cases} \Rightarrow x^2y = xy^2.$$

It follows that $x=y$ and $2000 = x^3 = y^3$, just as before.

21. Let $S(m,b) = \sum_{i=1}^{n} (y_i - mx_i + b)^2$

$$\nabla S(m,b) = \left(-2\sum_{i=1}^{n}(y_i - mx_i + b)x_i,\ 2\sum_{i=1}^{n}(y_i - mx_i + b)\right)$$

$$H_{S(m,b)} = \begin{pmatrix} \sum_{i=1}^{n} 2x_i^2 & \sum_{i=1}^{n} -2x_i \\ \sum_{i=1}^{n} -2x_i & 2n \end{pmatrix} = 2\begin{pmatrix} \sum_{i=1}^{n} x_i^2 & -\sum_{i=1}^{n} x_i \\ -\sum_{i=1}^{n} x_i & n \end{pmatrix}$$

Observe that $\det\left(2\sum_{i=1}^{n} x_i^2\right) > 0$ and $\det\left(H_{S(m,b)}\right) =$
$2^2\left(n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right)$.

To see that $\det(H_{S(m,b)}) > 0$ observe that $\sum_{i=1}^{n} x_i =$
$= (1,1,1,\dots,1)\cdot(x_1,x_2,x_3,\dots,x_n) \leq \|(1,1,1,\dots,1)\|\,\|(x_1,x_2,x_3,\dots,x_n)\|$
$= \sqrt{n}\left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$. Thus $\left(\sum_{i=1}^{n} x_i\right)^2 = \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i\right) <$
$< \sqrt{n}\left(\sum_{i=1}^{n} x_i^2\right)^{1/2}\sqrt{n}\left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = n\sum_{i=1}^{n} x_i^2$.

Thus $n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2 > 0$ as desired.

(what inequalities are we using and what allows us to assume that these inequalities are strict?)

It follows that $H_{S(m,b)}$ is positive-definite for any choice (m,b). In particular, any critical point is a local min.

Notice now that $\nabla S(m,b)=0$ whenever

$$\begin{cases} \sum_{i=1}^{n} (y_i - mx_i + b)\, x_i = 0 \\ \sum_{i=1}^{n} (y_i - mx_i + b) = 0 \end{cases}$$

This system of equations reduces to

$$(1) \quad \begin{cases} \left(\sum_{i=1}^{n} x_i^2\right) m - \left(\sum_{i=1}^{n} x_i\right) b = \sum_{i=1}^{n} x_i y_i \\ \left(\sum_{i=1}^{n} x_i\right) m - n b = \sum_{i=1}^{n} y_i \end{cases}$$

Or, in matrix form

$$(2) \quad \begin{pmatrix} \sum_{i=1}^{n} x_i^2 & -\sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & -n \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{pmatrix}$$

Thus $$\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} x_i^2 & -\sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & -n \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -n & \sum_{i=1}^{n} x_i \\ -\sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{pmatrix} \frac{1}{\left(\sum_{i=1}^{n} x_i\right)^2 - n\sum_{i=1}^{n} x_i^2} \begin{pmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{pmatrix}$$

Thus $m = \dfrac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) - n\sum_{i=1}^{n} x_i y_i}{\left(\sum_{i=1}^{n} x_i\right)^2 - n\sum_{i=1}^{n} x_i^2} =$

$$= \frac{\dfrac{\sum_{i=1}^{n} x_i}{n} \dfrac{\sum_{i=1}^{n} y_i}{n} - \dfrac{1}{n}\sum_{i=1}^{n} x_i y_i}{\left(\sum_{i=1}^{n} \dfrac{x_i}{n}\right)^2 - \dfrac{1}{n}\sum_{i=1}^{n} x_i^2} =$$

$$= \frac{\bar{x}\,\bar{y} - 1/n \sum_{i=1}^{n} x_i y_i}{\bar{x}^2 - 1/n \sum_{i=1}^{n} x_i^2} = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}$$

Observe that the bottom equation of (2) implies that

$b = \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right) m - \frac{1}{n}\left(\sum_{i=1}^{n} y_i\right) = \bar{x}\,m - \bar{y}$

For these values of (m, b), $S(m,b)$ attains a minimum.